

PROPERTIES OF PARABOLIC SOBOLEV AND PARABOLIC BESOV SPACES

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ABSTRACT. In this paper, we characterize parabolic Besov and parabolic Sobolev spaces in \mathbf{R}^{n+1} and \mathbf{R}_T^{n+1} , $T > 0$. We also, study the relation between parabolic Besov spaces in \mathbf{R}_T^n , $T > 0$ and standard Besov space in \mathbf{R}^n .

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1. INTRODUCTION

In this paper, we study the properties of the parabolic Besov spaces $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ and the parabolic Sobolev spaces $\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ for $1 \leq p \leq \infty$, $\alpha \in \mathbf{R}$. We also, study the relation between the parabolic Besov spaces in $\mathbf{R}_T^n = \{(X, t) | X \in \mathbf{R}^n, 0 < t < T\}$, $0 < T \leq \infty$ and the standard Besov space in \mathbf{R}^n .

The parabolic Sobolev spaces were studied in [9] and [17]. The authors in [9] proved the trace theorem of parabolic functions which is similar to the usual Sobolev spaces (see [2]). In fact, the parabolic Besov and the parabolic Sobolev spaces are particular case of the anisotropic Sobolev spaces and anisotropic Besov spaces, respectively, with dilation matrix $\delta_\epsilon = (\epsilon^2, \epsilon, \dots, \epsilon)$ (see [6], [7], [14] and [15]).

For the properties about the usual Besov and Sobolev spaces, we refer [2], [15], [16], [18], [20] and the references therein.

The Besov and Sobolev spaces have being used in boundary value problems of several elliptic type partial differential equations in bounded domain in \mathbf{R}^n . When boundary data is given with the function in some Besov or Sobolev spaces, one can find the solutions of the boundary value problems of elliptic type partial differential equations which contained in the corresponding spaces with boundary data (see [4], [5], [8], [11]).

Like the Besov and Sobolev spaces, functions in parabolic Besov and Sobolev spaces can be used with boundary data and solutions of initial-boundary value problems of parabolic type partial differential equations in bounded cylinder (see [12] for the case heat equation).

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In section 2, we introduce a parabolic Sobolev space $\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ and parabolic Besov space $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$. The properties to the parabolic Sobolev and parabolic Besov spaces are also stated.

In section 3, we show that $f \in \mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$, $1 < p < \infty$, $\alpha \in \mathbf{R}$ is equivalent to $f, D_{X_k}f, D_t^{\frac{1}{2}}f \in \mathcal{L}_{\alpha-1}^p(\mathbf{R}^{n+1})$ for all $1 \leq k \leq n$ (see Theorem 3.1), and that $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$, $1 \leq p \leq \infty$, $\alpha \in \mathbf{R}$ is equivalent to $f, D_{X_k}f, D_t^{\frac{1}{2}}f \in \mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}(\mathbf{R}^{n+1})$ for all $1 \leq i \leq n$ (see Theorem 3.4). Here $D_t^{\frac{1}{2}}$ is a fractional differential operator whose Fourier transform in term of t variable is defined by $\widehat{D_t^{\frac{1}{2}}f}(X, \tau) = |\tau|^{\frac{1}{2}}\widehat{f}(X, \tau)$.

Our result in section 3 can be compared with the results of V. Gopala Rao and B. Frank Jones. In [17], V. Gopala Rao showed that $f \in \mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ is equivalent to $f, D_t(f * h_1) \in \mathcal{L}_{\alpha-1}^p(\mathbf{R}^{n+1})$, where $*$ is a convolution in \mathbf{R}^{n+1} and $h_1(X, t) = c_1 t^{-\frac{n-1}{2}} e^{-\frac{|X|^2}{4t}}$ if $t > 0$ and $h_1(X, t) = 0$ if $t < 0$. In [10], B. Frank Jones induced several equivalent norms of parabolic Besov spaces. He also showed that $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$, $1 \leq p \leq \infty$, $\alpha \in \mathbf{R}$ is equivalent to $f, D_{X_k}f \in \mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}(\mathbf{R}^{n+1})$ and $D_t f \in \mathcal{B}_p^{\alpha-2, \frac{1}{2}\alpha-1}(\mathbf{R}^{n+1})$.

In section 4, we characterize the parabolic Besov spaces in $\mathbf{R}_T^n = \{(X, t) \in \mathbf{R}^{n+1} \mid 0 < t < T\}$, $0 < T \leq \infty$. We show that the parabolic Besov spaces in \mathbf{R}_T^n are also interpolation spaces and have the same properties as the Theorem 3.4.

In the section 5, we study the properties of the solution u of the heat equation with initial data $f \in \mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)$. We show that $u \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$, and we investigate an equivalent relation between the parabolic Besov norm $\|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)}$ and the usual Besov norm $\|f\|_{\mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)}$.

For $0 \leq \alpha$, $1 \leq p \leq \infty$ and $f \in \mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)$, we define a function by

$$(1.1) \quad u(X, t) = \langle f, \Gamma(X - \cdot, t) \rangle := \begin{cases} \langle f, \Gamma(X - \cdot, t) \rangle_\alpha, & 0 \leq \alpha < \frac{2}{p}, \\ \int_{\mathbf{R}^n} \Gamma(X - Y, t) f(Y) dY, & \frac{2}{p} \leq \alpha, \end{cases}$$

where $\Gamma(X, t) = c_n t^{-\frac{n}{2}} e^{-\frac{|X|^2}{4t}}$ if $t > 0$ and $\Gamma(X, t) = 0$ if $t < 0$, and $\langle \cdot, \cdot \rangle_\alpha$ is duality pairing between $\mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)$ and $\mathcal{B}_q^{-\alpha+\frac{2}{p}}(\mathbf{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that u is a solution to the heat equation in \mathbf{R}_∞^n with the initial value f . Our main result in section 5 are stated as follows.

Theorem 1.1. *Let $f \in \mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)$ and u be defined by (1.1). Let $1 \leq p \leq \infty$, $\alpha > 0$ and $T < \infty$. Then $u \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$ with*

$$\|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)} \approx \|f\|_{\mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)}.$$

The notation $A \approx B$ means that there are positive constants c and C independent of f such that $c \leq \frac{A}{B} \leq C$. Our result can be compared with the result of H. Triebel. In section 1.8.1 in [22], H. Triebel showed that for $1 < p < \infty$ and $\alpha > \frac{2}{p}$ and $m > \frac{1}{2}(\alpha - \frac{2}{p})$,

$$\|f\|_{L^p(\mathbf{R}^n)}^p + \int_0^\infty \int_{\mathbf{R}^n} t^{mp - \frac{1}{2}p(\alpha - \frac{2}{p})} |D_t^m u(X, t)|^p dX dt \approx \|f\|_{\mathcal{B}_p^{\alpha - \frac{2}{p}}(\mathbf{R}^n)}^p.$$

In this paper, we denote that $A \lesssim B$ means that $A \leq cB$ for positive constant c depending only on n, p , and T . We denote $\hat{\cdot}$ as the Fourier transform in \mathbf{R} , \mathbf{R}^n or \mathbf{R}^{n+1} .

2. PARABOLIC SOBOLEV AND PARABOLIC BESOV SPACES ON \mathbf{R}^{n+1}

For $\alpha \in \mathbf{R}$, we consider a distribution $H_\alpha(\xi, \tau)$ whose Fourier transform in \mathbf{R}^{n+1} is defined by

$$\widehat{H}_\alpha(\xi, \tau) = c_\alpha (1 + 4\pi^2 |\xi|^2 + i\tau)^{-\frac{\alpha}{2}}, \quad (\xi, \tau) \in \mathbf{R}^n \times \mathbf{R}.$$

For $\alpha \in \mathbf{R}$, $1 \leq p \leq \infty$, we define the parabolic Sobolev space $\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ by

$$\mathcal{L}_\alpha^p(\mathbf{R}^{n+1}) = \{f \in \mathcal{S}'(\mathbf{R}^{n+1}) \mid f = H_\alpha * g, \text{ for some } g \in L^p(\mathbf{R}^{n+1})\}$$

with norm

$$\|f\|_{\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})} := \|g\|_{L^p(\mathbf{R}^{n+1})} (= \|H_{-\alpha} * f\|_{L^p(\mathbf{R}^{n+1})}),$$

where $*$ is a convolution in \mathbf{R}^{n+1} and $\mathcal{S}'(\mathbf{R}^{n+1})$ is the dual space of the Schwartz space $\mathcal{S}(\mathbf{R}^{n+1})$. In particular, when $\alpha = 0$, we have that $\mathcal{L}_0^p(\mathbf{R}^{n+1}) = L^p(\mathbf{R}^{n+1})$.

Next, we define a parabolic Besov space. Let $\phi \in \mathcal{S}(\mathbf{R}^{n+1})$ such that

$$\begin{cases} \hat{\phi}(\xi, \tau) > 0 & \text{on } 2^{-1} < |\xi| + |\tau|^{\frac{1}{2}} < 2, \\ \hat{\phi}(\xi, \tau) = 0 & \text{elsewhere,} \\ \sum_{-\infty < i < \infty} \hat{\phi}(2^{-i}\xi, 2^{-2i}\tau) = 1 & ((\xi, \tau) \neq (0, 0)). \end{cases}$$

We define functions $\phi_i, \psi \in \mathcal{S}(\mathbf{R}^{n+1})$ whose Fourier transforms are written by

$$(2.1) \quad \begin{aligned} \hat{\phi}_i(\xi, \tau) &= \hat{\phi}(2^{-i}\xi, 2^{-2i}\tau) \quad (i = 0, \pm 1, \pm 2, \dots) \\ \hat{\psi}(\xi, \tau) &= 1 - \sum_{i=1}^{\infty} \hat{\phi}(2^{-i}\xi, 2^{-2i}\tau). \end{aligned}$$

Note that $\phi_i = 2^{(i+2)n} \phi(2^i X, 2^{2i} t)$. For $\alpha \in \mathbf{R}$ we define the parabolic Besov space $\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ by

$$\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1}) = \{f \in \mathcal{S}'(\mathbf{R}^{n+1}) \mid \|f\|_{\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}} < \infty\}$$

with the norms

$$\begin{aligned} \|f\|_{\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}} &:= \|\psi * f\|_{L^p} + \left(\sum_{1 \leq i < \infty} (2^{\alpha i} \|\phi_i * f\|_{L^p})^q \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, \\ \|f\|_{\mathcal{B}_{p\infty}^{\alpha, \frac{1}{2}\alpha}} &:= \sup(\|\psi * f\|_{L^p}, 2^{\alpha i} \|\phi_i * f\|_{L^p}), \end{aligned}$$

where $*$ is a convolution in \mathbf{R}^{n+1} . When $p = q$, we simply denote $\mathcal{B}_{pp}^{\alpha, \frac{1}{2}\alpha}$ by $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}$.

The following properties can be shown by the same argument as for the usual Sobolev space and the Besov space in \mathbf{R}^n .

Proposition 2.1. (1) *The definition of $\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ does not depend on the choice of the function ϕ ,*

(2) *The real interpolation method gives*

$$(\mathcal{L}_{\alpha_0}^p, \mathcal{L}_{\alpha_1}^p)_{\theta, q} = \mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}$$

for $1 \leq p \leq \infty$, $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $0 < \theta < 1$, and

$$(\mathcal{B}_{pq_0}^{\alpha_0, \frac{1}{2}\alpha_0}, \mathcal{B}_{pq_1}^{\alpha_1, \frac{1}{2}\alpha_1})_{\theta, r} = \mathcal{B}_{pr}^{\alpha, \frac{1}{2}\alpha}$$

for $\alpha_0 \neq \alpha_1$, $1 \leq p, r, q_0, q_1 \leq \infty$, $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$.

(3) *For $0 < \alpha < 2$, the the parabolic Besov norm $\|f\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}}$ is equivalent to the norm*

$$(2.2) \quad \|f\|_{L^p} + \left(\int_{\mathbf{R}^n} \int_{\mathbf{R} \times \mathbf{R}} \frac{|f(X, t) - f(X, s)|^p}{|t - s|^{1 + \frac{1}{2}p\alpha}} dt ds dX \right)^{\frac{1}{p}} \\ + \left(\int_{\mathbf{R}} \int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{|f(X + Y, t) - 2f(X, t) - f(X - Y, t)|^p}{|Y|^{n + p\alpha}} dY dX dt \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$;

$$(2.3) \quad \|f\|_{L^\infty} + \sup_{X, t, s, t \neq s} \frac{|f(X, t) - f(X, s)|}{|t - s|^{\frac{1}{2}p\alpha}} \\ + \sup_{t, s, X, Y, Y \neq 0} \frac{|f(X + Y, t) - 2f(X, t) - f(X - Y, t)|^p}{|Y|^{p\alpha}}.$$

if $p = \infty$.

(4) *The operator $S_\theta : \mathcal{L}_\alpha^p \rightarrow \mathcal{L}_{\alpha+\theta}^p$, $S_\alpha f = H_\alpha * f$ is isomorphism for all $\alpha, \theta \in \mathbf{R}$ and $1 \leq p \leq \infty$.*

(5) *$\mathcal{S}(\mathbf{R}^{n+1})$ is dense subset of $\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ for all $\alpha \in \mathbf{R}$ and $1 \leq p \leq \infty$.*

(6) *$\mathcal{L}_{\alpha_1}^p(\mathbf{R}^{n+1}) \subset \mathcal{L}_{\alpha_2}^p(\mathbf{R}^{n+1})$ for $\alpha_2 < \alpha_1$.*

For the details of the proof of Proposition 2.1 we refer [2] for (2) (in particular Definition 6.2.2, Theorem 6.2.4 and Theorem 6.4.5 in [2]), and refer [7] (Theorem 3) for (3). It is not difficult to derive (4) -(6) (see [2]).

For the sake of later use, we define $L^p(\mathbf{R}^n)$ -multiplier ($L^p(\mathbf{R}^{n+1})$ -multiplier) as follows.

Definition 2.2. *We say that $\mu \in \mathcal{S}'(\mathbf{R}^n)$ is $L^p(\mathbf{R}^n)$ -multiplier if*

$$(2.4) \quad \|\mathcal{F}^{-1}(\mu \hat{f})\|_{L^p(\mathbf{R}^n)} \leq M \|f\|_{L^p(\mathbf{R}^n)}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$, where $\mathcal{F}^{-1}(f)$ is the inverse Fourier transform of f . We call the minimal constant M satisfying (2.4) L^p -multiplier norm of μ .

Similarly, we define $L^p(\mathbf{R}^{n+1})$ -multiplier. We introduce the Marcinkiewicz multiplier theorem (see Theorem 4.6' in [18]).

Proposition 2.3. *Let m be a bounded function on $\mathbf{R}^n \setminus \{0\}$. Suppose also*

$$(a) \quad |\mu(\xi)| \leq B,$$

$$(b) \quad \text{for each } 0 < k \leq n,$$

$$\sup_{\xi_{k+1}, \dots, \xi_n} \int_{\rho} \left| \frac{\partial^k \mu}{\partial \xi_1 \partial \xi_2 \cdots \partial \xi_k} \right| d\xi_1 \cdots d\xi_k \leq B$$

as ρ ranges over dyadic rectangles of \mathbf{R}^k (If $k = n$, the "sup" sign is omitted).

$$(c) \quad \text{The condition analogous to (b) is valid for every one of the } n! \text{ permutations of the variables } \xi_1, \xi_2, \dots, \xi_n.$$

Then μ is L^p -multiplier, $1 < p < \infty$ and the multiplier norm depend only on B , p and n .

We denote by $D_{X_k}^i$, $i \in \mathbf{N} \cup \{0\}$ the i times derivatives with respect to X_k . When $i = 1$, we denote $D_{X_k}^1 = D_{X_k}$. We also denote D_X^β by the $D_{X_1}^{\beta_1} \cdots D_{X_n}^{\beta_n}$ for $\beta \in (\mathbf{N} \cup \{0\})^n$. We denote by $D_t^{\frac{1}{2}}$ the pseudo-differential operator whose Fourier transform is defined by $\widehat{D_t^{\frac{1}{2}} f}(\tau) = |\tau|^{\frac{1}{2}} \hat{f}(\tau)$ for complex-valued function f . It is well-known that

$$(2.5) \quad D_t^{\frac{1}{2}} f(t) = c \int_{\mathbf{R}} \frac{f(t) - f(s)}{|t - s|^{\frac{3}{2}}} ds$$

for complex-value function f . For non-negative integer, we also denote $D_t^i f$ by i times derivatives of f and $D_t^{i+\frac{1}{2}} f$ by $D_t^{\frac{1}{2}} D_t^i f$, respectively. Note that $D_t f = H D_t^{\frac{1}{2}} D_t^{\frac{1}{2}} f$.

3. THE PROPERTIES OF PARABOLIC SOBOLEV AND PARABOLIC BESOV SPACES

In this section, we study the properties of parabolic Sobolev and parabolic Besov spaces.

Theorem 3.1. *Let $1 < p < \infty$ and $\alpha \in \mathbf{R}$. Then $f \in \mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ if and only if $f, D_{X_k} f, D_t^{\frac{1}{2}} f \in \mathcal{L}_{\alpha-1}^p(\mathbf{R}^{n+1})$ for all $1 \leq k \leq n$. Furthermore,*

$$(3.1) \quad \|f\|_{\mathcal{L}_\alpha^p} \approx \|f\|_{\mathcal{L}_{\alpha-1}^p} + \sum_{1 \leq k \leq n} \|D_{X_k} f\|_{\mathcal{L}_{\alpha-1}^p} + \|D_t^{\frac{1}{2}} f\|_{\mathcal{L}_{\alpha-1}^p}.$$

Proof. First, we assume $\alpha = 1$. Suppose $f \in \mathcal{L}_1^p(\mathbf{R}^{n+1})$ so that $f = H_1 * g$ for some $g \in L^p(\mathbf{R}^{n+1})$. Then, for $1 \leq k \leq n$, we have

$$(3.2) \quad \widehat{D_{X_k} f} = \frac{-2\pi\xi_k}{(1+4\pi^2|\xi|^2+i\tau)^{\frac{1}{2}}} \hat{g}, \quad \widehat{D_t^{\frac{1}{2}} f} = \frac{|\tau|^{\frac{1}{2}}}{(1+4\pi^2|\xi|^2+i\tau)^{\frac{1}{2}}} \hat{g}.$$

Applying Proposition 2.3, we have that $\mu_{1k}(\xi, \tau) = \frac{-2\pi\xi_k}{(1+4\pi^2|\xi|^2+i\tau)^{\frac{1}{2}}}$, $\mu_2(\xi, \tau) = \frac{|\tau|^{\frac{1}{2}}}{(1+4\pi^2|\xi|^2+i\tau)^{\frac{1}{2}}}$ are $L^p(\mathbf{R}^{n+1})$ multipliers for $1 < p < \infty$. Then, from (3.2), we get

$$\begin{aligned} \|D_{X_k} f\|_{L^p} &= \|\mathcal{F}^{-1}(\mu_{1k}(\xi, \tau)\hat{g})\|_{L^p} \lesssim \|g\|_{L^p} = \|f\|_{\mathcal{L}_1^p} \quad 1 \leq k \leq n, \\ \|D_t^{\frac{1}{2}} f\|_{L^p} &= \|\mathcal{F}^{-1}(\mu_2(\xi, \tau)\hat{g})\|_{L^p} \lesssim \|g\|_{L^p} = \|f\|_{\mathcal{L}_1^p}. \end{aligned}$$

From (6) in Proposition 2.1, we obtain $\|f\|_{L^p} \lesssim \|f\|_{\mathcal{L}_1^p}$. Hence, we proved the one-side of Theorem 3.1.

Now, we prove the converse inequality. Suppose $f, D_t^{\frac{1}{2}} f, D_{X_k} f \in L^p(\mathbf{R}^{n+1})$, $1 \leq k \leq n$. We claim that $f = H_1 * g$ for some $g \in L^p(\mathbf{R}^{n+1})$ satisfying

$$(3.3) \quad \|g\|_{L^p} \lesssim (\|f\|_{L^p} + \sum_{1 \leq k \leq n} \|D_{X_k} f\|_{L^p} + \|D_t^{\frac{1}{2}} f\|_{L^p}).$$

If then, $f = H_1 * g \in \mathcal{L}_1^p(\mathbf{R}^{n+1})$ with $\|f\|_{\mathcal{L}_1^p} \lesssim (\|f\|_{L^p} + \sum_{1 \leq k \leq n} \|D_{X_k} f\|_{L^p} + \|D_t^{\frac{1}{2}} f\|_{L^p})$, and this will complete the proof of Theorem 3.1.

To prove the claim, let us R_k , $1 \leq k \leq n$ be Riesz transforms in \mathbf{R}^n . Then, we have

$$\mathcal{F}^{-1}((1 + |\xi| + |\tau|^{\frac{1}{2}})\hat{f}) = f + \sum_{1 \leq k \leq n} R_k \frac{\partial f}{\partial x_k} + D_t^{\frac{1}{2}} f \in L^p(\mathbf{R}^{n+1}).$$

Set $\hat{K}(\xi, \tau) = \frac{(1+4\pi^2|\xi|^2+i\tau)^{\frac{1}{2}}}{1+|\xi|+|\tau|^{\frac{1}{2}}}$ and $g = K * (f + \sum_{1 \leq k \leq n} R_k \frac{\partial f}{\partial x_k} + D_t^{\frac{1}{2}} f)$. Applying Proposition 2.3, we have that $\hat{K}(\xi, \tau)$ is $L^p(\mathbf{R}^{n+1})$ -multiplier. Hence we have $g \in L^p(\mathbf{R}^{n+1})$. Hence, (3.1) holds for $\alpha = 1$.

For general $\alpha \in \mathbf{R}$, by (4) in Proposition 2.1, we have that $S_{\alpha-1} : \mathcal{L}_1^p \rightarrow \mathcal{L}_\alpha^p$ and $S_{\alpha-1} : L^p \rightarrow \mathcal{L}_{\alpha-1}^p$ are isomorphism whose inverses are $S_{\alpha-1}^{-1} = S_{-\alpha+1}$. Note that $D_{X_k} S_{-\alpha+1} f = S_{-\alpha+1} D_{X_k} f$ and $D_t^{\frac{1}{2}} S_{-\alpha+1} f = S_{-\alpha+1} D_t^{\frac{1}{2}} f$. Hence, we get

$$\begin{aligned} f \in \mathcal{L}_\alpha^p &\Leftrightarrow S_{\alpha-1}^{-1} f = S_{-\alpha+1} f \in \mathcal{L}_1^p \\ &\Leftrightarrow S_{-\alpha+1} f, D_{X_k} S_{-\alpha+1} f (= S_{-\alpha+1} D_{X_k} f), D_t^{\frac{1}{2}} S_{-\alpha+1} f (= S_{-\alpha+1} D_t^{\frac{1}{2}} f) \in L^p \\ &\Leftrightarrow f, D_{X_k} f, D_t^{\frac{1}{2}} f \in \mathcal{L}_{\alpha-1}^p. \end{aligned}$$

Hence, we complete the proof of (3.1). \square

Corollary 3.2. *Let $1 < p < \infty$ and $\alpha \in \mathbf{R}$. Then $f \in \mathcal{L}_2^p(\mathbf{R}^{n+1})$ if and only if $f, D_{X_k} D_{X_l} f, D_t f \in L^p(\mathbf{R}^{n+1})$ for all $1 \leq k, l \leq n$. Furthermore,*

$$(3.4) \quad \|f\|_{\mathcal{L}_\alpha^p} \approx \|f\|_{\mathcal{L}_{\alpha-2}^p} + \sum_{0 \leq k, l \leq n} \|D_{X_k} D_{X_l} f\|_{\mathcal{L}_{\alpha-2}^p} + \|D_t f\|_{\mathcal{L}_{\alpha-2}^p}.$$

Proof. As the proof of Theorem 3.1, it suffices to show the Corollary when $\alpha = 2$. Suppose $f \in \mathcal{L}_2^p(\mathbf{R}^{n+1})$. Since $\mathcal{L}_0^p(\mathbf{R}^{n+1}) = L^p(\mathbf{R}^{n+1})$, applying the Theorem 3.1 two times, we have

$$(3.5) \quad \begin{aligned} \|f\|_{\mathcal{L}_2^p} &\approx \|f\|_{L^p} + \sum_{0 \leq k \leq n} \|D_{X_k} f\|_{L^p} + \|D_t^{\frac{1}{2}} f\|_{L^p} + \sum_{0 \leq k, l \leq n} \|D_{X_k} D_{X_l} f\|_{L^p} \\ &\quad + \sum_{0 \leq k \leq n} \|D_t^{\frac{1}{2}} D_{X_k} f\|_{L^p} + \|D_t f\|_{L^p}. \end{aligned}$$

Hence, if $f \in \mathcal{L}_2^p(\mathbf{R}^{n+1})$, then we have

$$\|f\|_{L^p} + \sum_{0 \leq k, l \leq n} \|D_{X_k} D_{X_l} f\|_{L^p} + \left\| \frac{\partial f}{\partial t} \right\|_{L^p} \lesssim \|f\|_{\mathcal{L}_2^p}.$$

Conversely, suppose that $f, D_{X_k} D_{X_l} f, D_t f \in L^p(\mathbf{R}^{n+1})$. Note that applying Proposition 2.3, we have that $\nu_1(\xi, \tau) = \frac{|\tau|^{\frac{1}{2}}}{1+4\pi^2|\xi|^2+i\tau}$, $\nu_{2,k}(\xi, \tau) = \frac{2\pi i \xi_k}{1+4\pi^2|\xi|^2+i\tau}$, $\nu_{3,k}(\xi, \tau) = \frac{2\pi i \xi_k |\tau|^{\frac{1}{2}}}{1+4\pi^2|\xi|^2+i\tau}$, $1 < p < \infty$ are $L^p(\mathbf{R}^{n+1})$ -multipliers. Then, we have

$$(3.6) \quad \begin{aligned} \widehat{D_t^{\frac{1}{2}} f} &= \nu_1(\xi, \tau)(1 + 4\pi^2|\xi|^2 + i\tau)\hat{f}, \quad \widehat{D_{X_k} f} = \nu_{2,k}(\xi, \tau)(1 + 4\pi^2|\xi|^2 + i\tau)\hat{f}, \\ \widehat{D_t^{\frac{1}{2}} D_{X_k} f} &= \nu_{3,k}(\xi, \tau)(1 + 4\pi^2|\xi|^2 + i\tau)\hat{f}. \end{aligned}$$

Note that $\mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2 + i\tau)\hat{f}) = f + \sum_{1 \leq k \leq n} D_{X_k}^2 f + D_t f$. Hence, from (3.6), we have

$$(3.7) \quad \|D_t^{\frac{1}{2}} f\|_{L^p} + \|D_{X_k} f\|_{L^p} + \|D_t^{\frac{1}{2}} D_{X_k} f\|_{L^p} \lesssim (\|f\|_{L^p} + \|D_t f\|_{L^p} + \sum_{1 \leq k, l \leq n} \|D_{X_k} D_{X_l} f\|_{L^p}).$$

With (3.5), (3.7) and the assumption, this implies

$$\|f\|_{\mathcal{L}_2^p} \lesssim (\|f\|_{L^p} + \sum_{0 \leq k, l \leq n} \|D_{X_k} D_{X_l} f\|_{L^p} + \|D_t f\|_{L^p}).$$

Hence, we completed the proof of Corollary 3.2. \square

Now, we define parabolic Sobolev space $\tilde{W}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ and $W_p^{2\alpha, \alpha}(\mathbf{R}^{n+1})$ for positive integer α and $1 \leq p \leq \infty$ by

$$\begin{aligned} \tilde{W}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1}) &:= \{f \in L^p(\mathbf{R}^{n+1}) \mid D_X^\beta D_t^{\frac{l}{2}} f \in L^p(\mathbf{R}^{n+1}), \quad |\beta| + l \leq \alpha\}, \\ W_p^{2\alpha, \alpha}(\mathbf{R}^{n+1}) &:= \{f \in L^p(\mathbf{R}^{n+1}) \mid D_X^\beta D_t^l f \in L^p(\mathbf{R}^{n+1}), \quad |\beta| + 2l \leq 2\alpha\} \end{aligned}$$

with norms

$$\|f\|_{\tilde{W}_p^{\alpha, \frac{1}{2}\alpha}} := \sum_{|\beta| + \frac{1}{2}l \leq \alpha} \|D_X^\beta D_t^{\frac{l}{2}} f\|_{L^p}, \quad \|f\|_{W_p^{2\alpha, \alpha}} := \sum_{|\beta| + l \leq 2\alpha} \|D_X^\beta D_t^l f\|_{L^p}.$$

Remark 3.3. (1) From the Theorem 3.1 and Corollary 3.2, if α is non-negative integer and $1 < p < \infty$, then we have

$$(3.8) \quad \mathcal{L}_\alpha^p(\mathbf{R}^{n+1}) = \tilde{W}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1}), \quad \mathcal{L}_{2\alpha}^p(\mathbf{R}^{n+1}) = \tilde{W}_p^{2\alpha, \alpha}(\mathbf{R}^{n+1}) = W_p^{2\alpha, \alpha}(\mathbf{R}^{n+1})$$

with the equivalent norms.

(2) When $p = 1$ or $p = \infty$, the spaces $\mathcal{L}_\alpha^p(\mathbf{R}^{n+1})$ and $\tilde{W}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ are different spaces, and $\mathcal{L}_{2\alpha}^p(\mathbf{R}^{n+1})$, $\tilde{W}_p^{2\alpha, \alpha}(\mathbf{R}^{n+1})$ and $W_p^{2\alpha, \alpha}(\mathbf{R}^{n+1})$ are different spaces each other.

Next, we study about the properties of parabolic Besov spaces.

Theorem 3.4. Let $1 \leq p \leq \infty$ and $\alpha \in \mathbf{R}$. Then $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ if and only if $f, D_{X_k}f, D_t^{\frac{1}{2}}f \in \mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}(\mathbf{R}^{n+1})$ for all $1 \leq k \leq n$. Furthermore,

$$(3.9) \quad \|f\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}} \approx \|f\|_{\mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} + \sum_{1 \leq k \leq n} \|D_{X_k}f\|_{\mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} + \|D_t^{\frac{1}{2}}f\|_{\mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}}.$$

Proof. If $1 < p < \infty$, then by Theorem 3.1 and the property of interpolation spaces (see (2) of Proposition 2.1), (3.9) holds. Hence we have only to consider the critical case $p = 1$ and $p = \infty$. Since the proofs are exactly same, we only prove in the case of $p = 1$.

Suppose that $f \in \mathcal{B}_1^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$. Then by the definition of the parabolic Besov space, we have

$$\|f\|_{\mathcal{B}_1^{\alpha, \frac{1}{2}\alpha}} = \|f * \psi\|_{L^1} + \sum_{1 \leq i < \infty} 2^{\alpha i} \|f * \phi_i\|_{L^1} < \infty.$$

Note that by construction of ψ and ϕ_i in section 2, we have $\hat{\psi} + \hat{\phi}_1 + \hat{\phi}_2 = 1$ in $\text{supp}(\hat{\psi} + \hat{\phi}_1)$ and $\hat{\phi}_{i-1} + \hat{\phi}_i + \hat{\phi}_{i+1} = 1$ in $\text{supp} \hat{\phi}_i$ for $i \geq 2$. Hence, using $D_{X_k}(f * g) = (D_{X_k}f) * g = f * (D_{X_k}g)$, we have

$$\begin{aligned} (D_{X_k}f) * \psi &= f * \psi * D_{X_k}(\psi + \phi_1 + \phi_2), \\ (D_{X_k}f) * \phi_1 &= f * \phi_1 * D_{X_k}(\psi + \phi_1 + \phi_2), \\ (D_{X_k}f) * \phi_i &= f * \phi_i * D_{X_k}(\phi_{i-1} + \phi_i + \phi_{i+1}), \quad i \geq 2. \end{aligned}$$

Note that $\|D_{X_k}\psi\|_{L^1} \lesssim$ and $\|D_{X_k}\phi_i\|_{L^1} \lesssim 2^i$. Hence, by Young's inequality, we have

$$\begin{aligned} \|(D_{X_k}f) * \psi\|_{L^1} &\leq \|f * \psi\|_{L^1} \|D_{X_k}(\psi + \phi_1 + \phi_2)\|_{L^1} \lesssim \|f * \psi\|_{L^1}, \\ \|(D_{X_k}f) * \phi_1\|_{L^1} &\leq \|f * \phi_1\|_{L^1} \|D_{X_k}(\psi + \phi_1 + \phi_2)\|_{L^1} \lesssim \|f * \phi_1\|_{L^1}, \\ \|(D_{X_k}f) * \phi_i\|_{L^1} &\leq \|f * \phi_i\|_{L^1} \|D_{X_k}(\phi_{i-1} + \phi_i + \phi_{i+1})\|_{L^1} \lesssim 2^i \|f * \phi_i\|_{L^1}, \quad i \geq 2. \end{aligned}$$

Hence, we have

$$\begin{aligned}
\|D_{X_k} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} &= \|D_{X_k} f * \psi\|_{L^1} + \sum_{1 \leq i < \infty} 2^{(\alpha-1)i} \|D_{X_k} f * \phi_i\|_{L^1} \\
&\lesssim (\|f * \psi\|_{L^1} + \sum_{1 \leq i < \infty} 2^{\alpha i} \|f * \phi_i\|_{L^1}) \\
&= \|f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(D_t^{\frac{1}{2}} f) * \psi &= f * \psi * D_t^{\frac{1}{2}}(\psi + \phi_1 + \phi_2), \\
(D_t^{\frac{1}{2}} f) * \phi_1 &= f * \phi_1 * D_t^{\frac{1}{2}}(\psi + \phi_1 + \phi_2), \\
(D_t^{\frac{1}{2}} f) * \phi_i &= f * \phi_i * D_t^{\frac{1}{2}}(\phi_{i-1} + \phi_i + \phi_{i+1}), \quad i \geq 2.
\end{aligned}$$

Note that using (2.5) and change of variables, we have

$$\begin{aligned}
\|D_t^{\frac{1}{2}} \phi_i\|_{L^1} &= c \int_{\mathbf{R}^{n+1}} \left| \int_{\mathbf{R}} \frac{\phi_i(X, t) - \phi_i(X, s)}{|t-s|^{\frac{3}{2}}} ds \right| dX dt \\
&\lesssim 2^i \int_{\mathbf{R}^{n+1}} \int_{\mathbf{R}} \frac{|\phi(X, t) - \phi(X, s)|}{|t-s|^{\frac{3}{2}}} ds dX dt \\
&\lesssim 2^i \|\phi\|_{\mathcal{B}_1^{1, \frac{1}{2}}(\mathbf{R}^{n+1})} \\
\|D_t^{\frac{1}{2}} \psi\|_{L^1} &\lesssim \|\psi\|_{\mathcal{B}_1^{1, \frac{1}{2}}(\mathbf{R}^{n+1})}.
\end{aligned} \tag{3.10}$$

As the same reason to the case of $D_{X_k} f$, using Young's inequality, we have $\|D_t^{\frac{1}{2}} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} \lesssim \|f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}}$. Hence, we proved one side of (3.9).

Conversely, we suppose that $\|f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}}, \|D_{X_k} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}}, \|D_t^{\frac{1}{2}} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} < \infty$. Since ϕ is supported in $\{(\xi, \tau) \in \mathbf{R}^{n+1} \mid 2^{-1} < |\xi| + |\tau|^{\frac{1}{2}} < 2\}$, we have that $\frac{1}{(-4\pi^2|\xi|^2+i\tau)}\phi(\xi, \tau) \in \mathcal{S}(\mathbf{R}^{n+1})$. We define Φ and Φ_i by the functions whose Fourier transforms are written by $\hat{\Phi}(\xi, \tau) = \frac{1}{-4\pi^2|\xi|^2+i\tau}\phi(\xi, \tau)$ and $\hat{\Phi}_i(\xi, \tau) = \hat{\Phi}(2^{-i}\xi, 2^{-2i}\tau)$. Then, for $i \geq 2$, we have

$$\begin{aligned}
\widehat{f * \phi_i} &= \hat{f} \hat{\phi}_i(\hat{\phi}_{i-1} + \hat{\phi}_i + \hat{\phi}_{i+1}) \\
&= \hat{f} \hat{\phi}_i \frac{-4\pi^2|\xi|^2+i\tau}{-4\pi^2|\xi|^2+i\tau} (\hat{\phi}_{i-1} + \hat{\phi}_i + \hat{\phi}_{i+1}) \\
&= 2^{-2i} \sum_{1 \leq k \leq n} \widehat{D_{X_k} f} \hat{\phi}_i (\widehat{D_{X_k} \Phi_{i-1}} + \widehat{D_{X_k} \Phi_i} + \widehat{D_{X_k} \Phi_{i+1}}) \\
&\quad + 2^{-2i} \widehat{D_t^{\frac{1}{2}} f} \hat{\phi}_i (\widehat{H D_t^{\frac{1}{2}} \Phi_{i-1}} + \widehat{H D_t^{\frac{1}{2}} \Phi_i} + \widehat{H D_t^{\frac{1}{2}} \Phi_{i+1}}),
\end{aligned} \tag{3.11}$$

where H is the Hilbert transform. We used the fact that $D_t\Phi = HD_t^{\frac{1}{2}}D_t^{\frac{1}{2}}\Phi$. Note that $\|D_{X_k}\Phi_i\|_{L^1} \lesssim 2^i$. Moreover,

$$\begin{aligned} HD_t^{\frac{1}{2}}\Phi_i(X, t) &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon < |t-s| < \frac{1}{\epsilon}} \frac{\text{sign}(t-s)}{|t-s|^{\frac{3}{2}}} \Phi_i(X, s) ds \\ &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon < |t-s| < \frac{1}{\epsilon}} \frac{\text{sign}(t-s)}{|t-s|^{\frac{3}{2}}} (\Phi_i(X, s) - \Phi_i(X, t)) ds, \end{aligned}$$

where $\text{sign}(t) = 1$ if $t > 0$ and $\text{sign}(t) = -1$ if $t < 0$. Hence, using change of variables (see (3.10)), we get

$$\|HD_t^{\frac{1}{2}}\Phi_i\|_{L^1} \lesssim \int_{\mathbf{R}^{n+1}} \int_{\mathbf{R}} \frac{|\Phi_i(X, s) - \Phi_i(X, t)|}{|t-s|^{\frac{3}{2}}} ds dX dt \lesssim 2^i \|\Phi\|_{\mathcal{B}_1^{1, \frac{1}{2}}(\mathbf{R}^{n+1})}.$$

Hence, applying Young's inequality in (3.11), we have

$$(3.12) \quad \|f * \phi_i\|_{L^1} \leq 2^{-i} \left(\sum_{1 \leq k \leq n} \|D_{X_k} f * \phi_i\|_{L^1} + \|D_t^{\frac{1}{2}} f * \phi_i\|_{L^1} \right) \quad i \geq 2.$$

Hence by (3.12), we have

$$\begin{aligned} \|f\|_{\mathcal{B}_1^{\alpha, \frac{1}{2}\alpha}} &= \|f * \psi\|_{L^1} + \sum_{1 \leq i < \infty} 2^{\alpha i} \|f * \phi_i\|_{L^1} \\ &\lesssim \left(\|f * \psi\|_{L^1} + \|f * \phi_1\|_{L^1} + \sum_{2 \leq i < \infty} 2^{(\alpha-1)i} \left(\sum_{1 \leq k \leq n} \|D_{X_k} f * \phi_i\|_{L^1} + \|D_t^{\frac{1}{2}} f * \phi_i\|_{L^1} \right) \right) \\ &\lesssim \left(\|f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} + \|D_{X_k} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} + \|D_t^{\frac{1}{2}} f\|_{\mathcal{B}_1^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}} \right). \end{aligned}$$

Hence, we completed the proof of Theorem 3.4. \square

By (2.2), (2.3) and Theorem 3.4, we get the following Corollary;

Corollary 3.5. (1) Let $1 \leq p \leq \infty$ and $\alpha \in \mathbf{R}$ such that $2i < \alpha < 2i + 2$ for positive integer i . Then $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ if and only if $f, D_X^\beta f, D_t^i f \in \mathcal{B}_p^{\alpha-2i, \frac{1}{2}\alpha-i}(\mathbf{R}^{n+1})$ for all $|\beta| = 2i$. Furthermore,

$$\|f\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}} \approx \|f\|_{\mathcal{B}_p^{\alpha-2i, \frac{1}{2}\alpha-i}} + \sum_{|\beta|=2i} \|D_X^\beta f\|_{\mathcal{B}_p^{\alpha-2i, \frac{1}{2}\alpha-i}} + \|D_t^i f\|_{\mathcal{B}_p^{\alpha-2i, \frac{1}{2}\alpha-i}}.$$

(2) In particular, for $1 \leq p < \infty$, we have

$$\begin{aligned} \|f\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}}^p &\approx \|f\|_{W_p^{2i, i}}^p + \int_{\mathbf{R}^n} \int_{\mathbf{R} \times \mathbf{R}} \frac{|D_t^i f(X, t) - D_t^i f(X, s)|^p}{|t-s|^{1+p\frac{1}{2}(\alpha-2i)}} dt ds dX \\ &\quad + \sum_{|\beta|=2i} \int_{\mathbf{R}} \int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{|D_X^\beta f(X+Y, t) - 2D_X^\beta f(X, t) + D_X^\beta f(X-Y, t)|^p}{|Y|^{n+p(\alpha-2i)}} dX dY dt \end{aligned}$$

and

$$\begin{aligned} \|f\|_{\mathcal{B}_\infty^{\alpha, \frac{1}{2}\alpha}} &\approx \|f\|_{W_\infty^{2i, i}} + \sup_{X, t, s, t \neq s} \frac{|D_t^i f(X, t) - D_t^i f(X, s)|}{|t - s|^{\frac{1}{2}(\alpha - 2i)}} \\ &+ \sum_{|\beta|=2i} \sup_{t, s, X, Y, Y \neq 0} \frac{|D_X^\beta f(X + Y, t) - 2D_X^\beta f(X, t) - D_X^\beta f(X - Y, t)|}{|Y|^{\alpha - 2i}}. \end{aligned}$$

Proof. Applying Theorem 3.4 two times, we obtain one side of (1). To show that the right side of (2) implies the left side of (3), we replace $\frac{1}{-4\pi^2|\xi|^2+i\tau}\phi(\xi, \tau)$ by $\frac{1}{(-4\pi^2|\xi|^2)^i+(i\tau)^i}\phi(\xi, \tau)$ in (3.11) and apply the proof of Theorem 3.4. (2) holds because of (1) and (2.1). \square

4. PARABOLIC SOBOLEV AND PARABOLIC BESOV SPACE IN \mathbf{R}_T^n

If i is non-negative integer, we define the parabolic Sobolev space $W_p^{2i, i}(\mathbf{R}_T^n)$, $0 < T \leq \infty$ by

$$W_p^{2i, i}(\mathbf{R}_T^n) = \{f \mid D_X^\beta D_t^l f \in L^p(\mathbf{R}_T^n), 0 \leq |\beta| + 2l \leq 2i\},$$

so that the norm in $W_p^{2i, i}(\mathbf{R}_T^n)$ is defined by

$$\begin{aligned} \|f\|_{W_p^{2i, i}(\mathbf{R}_T^n)} &= \left(\sum_{2l+|\beta| \leq 2i} \int \int_{\mathbf{R}_T^n} |D_X^\beta D_t^l f(X, t)|^p dX dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \\ \|f\|_{W_\infty^{2i, i}(\mathbf{R}_T^n)} &= \sum_{2l+|\beta| \leq 2i} \sup_{(X, t) \in \mathbf{R}_T^n} |D_X^\beta D_t^l f(X, t)|, \quad p = \infty \end{aligned}$$

Let $2i < \alpha < 2i + 2$. We would like to define parabolic Besov space $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$. We say that $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$ if and only if

$$\begin{aligned} \|f\|_{W_p^{2i, i}(\mathbf{R}_T^n)}^p &+ \sum_{|\beta|+2l=2i} \left[\int_{\mathbf{R}^n} \int_0^T \int_0^T \frac{|D_X^\beta D_t^l f(X, t) - D_X^\beta D_t^l f(X, s)|^p}{|t - s|^{1+\frac{1}{2}p(\alpha-2i)}} dt ds dX \right. \\ &\left. \int_0^T \int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{|D_X^\beta D_t^l f(X + Y, t) - 2D_X^\beta D_t^l f(X, t) + D_X^\beta D_t^l f(X - Y, t)|^p}{|Y|^{n+p(\alpha-2i)}} dX dY dt \right] < \infty \end{aligned}$$

if $1 \leq p < \infty$ and

$$\begin{aligned} \|f\|_{W_\infty^{2i, i}(\mathbf{R}_T^n)} &+ \sum_{|\beta|+2l=2i} \left[\sup_{X, t, s, t \neq s} \frac{|D_X^\beta D_t^l f(X, t) - D_X^\beta D_t^l f(X, s)|}{|t - s|^{\frac{1}{2}(\alpha-2i)}} \right. \\ &\left. + \sup_{t, X, Y, Y \neq 0} \frac{|D_X^\beta D_t^l f(X + Y, t) - 2D_X^\beta D_t^l f(X, t) + D_X^\beta D_t^l f(X - Y, t)|}{|Y|^{\alpha-2i}} \right] < \infty. \end{aligned}$$

Proposition 4.1. *Let $1 \leq p \leq \infty$. Suppose that there is a bounded linear operator $E_{\mathbf{R}_T^n} : W_p^{2i, i}(\mathbf{R}_T^n) \rightarrow \mathcal{L}_{2i}^p(\mathbf{R}^{n+1})$ for all non-negative integer i and $1 \leq p \leq \infty$ such that $E_{\mathbf{R}_T^n} f = f$ in \mathbf{R}_T^n . Then for $0 < \theta < 1$, $i < l$, we get $(W_p^{2i, i}(\mathbf{R}_T^n), W_p^{2l, l}(\mathbf{R}_T^n))_{p, \theta} = \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$, where $\alpha = (1 - \theta)2i + \theta 2l$.*

Proof. Applying Theorem 4.12 and Corollary 4.13 in [4], for $i \in \mathbf{N}$, $1 \leq p \leq \infty$ and $0 < \theta < 1$, we obtain that $(L^p(\mathbf{R}^{n+1}), W_p^{2i,i}(\mathbf{R}^{n+1}))_{\theta p} = \mathcal{B}_p^{2i\theta, i\theta}(\mathbf{R}^{n+1})$. Using the Proposition 2.4 and the Proposition 2.17 in [11], we obtain Proposition 4.1. \square

To apply the Proposition 4.1, we define extension operators from $W_p^{2i,i}(\mathbf{R}_\infty^n)$ to $\mathcal{L}_{2i}^p(\mathbf{R}^{n+1})$ and from $W_p^{2i,i}(\mathbf{R}_T^n)$ to $\mathcal{L}_{2i}^p(\mathbf{R}^{n+1})$. For $f \in W_p^{2i,i}(\mathbf{R}_\infty^n)$ we define extension E_2f of f by

$$(4.1) \quad E_2f(X, t) = \begin{cases} f(X, t), & t \geq 0, \\ \sum_{1 \leq j \leq 2i+1} \lambda_j f(X, -jt) & t \leq 0, \end{cases}$$

where the coefficients $\lambda_1, \dots, \lambda_{2i+1}$ are the unique solution of the $(2i+1) \times (2i+1)$ system of linear equations

$$\sum_{1 \leq j \leq 2i+1} (-j)^l \lambda_j = 1, \quad l = 0, 1, \dots, 2i.$$

Then $E_2f \in W_p^{2i,i}(\mathbf{R}^{n+1})$ with $E_2f|_{\mathbf{R}_\infty^n} = f$, $\|E_2f\|_{W_p^{2i,i}(\mathbf{R}^{n+1})} \leq c\|f\|_{W_p^{2i,i}(\mathbf{R}_\infty^n)}$ (see Theorem 4.26 in [1]).

We apply (4.1) to define the extension operator in $W_p^{2i,i}(\mathbf{R}_T^n)$. Let $g \in W_p^{2i,i}(\mathbf{R}_T^n)$. We define an extension E_3 by

$$E_3g(X, t) = \theta(t) \begin{cases} \sum_{1 \leq j \leq 2i+1} \lambda_j g(X, -jt) & -T < t < 0, \\ g(X, t) & 0 < t < T, \\ \sum_{1 \leq j \leq 2i+1} \lambda_j g(X, -j(2T-t)) & T < t < 2T. \end{cases}$$

and $E_3g(X, t) = 0$ otherwise, where $\theta \in C_c^\infty(\mathbf{R})$ such that $\theta \equiv 1$ in $(0, T)$ and $\text{supp } \theta \subset (-T, 2T)$. Then $E_3g|_{\mathbf{R}_T^n} = g$ and $\|E_3g\|_{W_p^{2i,i}(\mathbf{R}^{n+1})} \lesssim \|g\|_{W_p^{2i,i}(\mathbf{R}_T^n)}$.

By Proposition 4.1, we have the following theorem.

Theorem 4.2. *Then, for $0 < \alpha$ and $1 \leq p \leq \infty$, $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$ is real interpolation space, that is, $(L^p(\mathbf{R}_T^n), W_p^{2i,i}(\mathbf{R}_T^n))_{p, \theta} = \mathcal{B}_p^{2(1-\theta)i, (1-\theta)i}(\mathbf{R}_T^n)$, $0 < T \leq \infty$.*

Theorem 4.3. *Then, for $\alpha \geq 2$, and $1 \leq p \leq \infty$, $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$ if and only if $f, D_{X_k}f, D_{X_k}D_{X_j}f, D_tf \in \mathcal{B}_p^{\alpha-2, \frac{1}{2}\alpha-1}(\mathbf{R}_T^n)$, $0 < T \leq \infty$.*

Proof. Because of the similarity of the proof, we consider only the case of \mathbf{R}_∞^n . We define extension operator,

$$E_4f(X, t) = \begin{cases} f(X, t) & t > 0, \\ \sum_{1 \leq j \leq 2i+1} (-j) \lambda_j f(X, -jt) & t < 0. \end{cases}$$

Then, $E_4 : W_p^{2l-2, l-1}(\mathbf{R}_\infty^n) \rightarrow W_p^{2l-2, l-1}(\mathbf{R}^{n+1})$, $0 \leq l \leq i$ is bounded operator and so by (4.2), we get $E_4 : \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_\infty^n) \rightarrow \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$, $\alpha > 0$, $1 \leq p \leq \infty$ is bounded operator. Note that

$$(4.2) \quad D_{X_k}(E_2f) = E_2(D_{X_k}f), \quad D_{X_i}D_{X_k}(E_2f) = E_2(D_{X_i}D_{X_k}f), \quad D_t(E_2f) = E_4(D_tf).$$

Let $f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_+^{n+1})$. Then $E_2 f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$ and by Corollary 3.2, we have

$$E_2 f, D_{X_k}(E_2 f), D_{X_i X_k}(E_2 f), D_t(E_2 f) \in \mathcal{B}_p^{\alpha-2, \frac{1}{2}\alpha-1}(\mathbf{R}^{n+1}).$$

Hence by (4.2), we have

$$(4.3) \quad f, D_{X_k} f, D_{X_i} D_{X_k} f, D_t f \in \mathcal{B}_p^{\alpha-2, \frac{1}{2}\alpha-1}(\mathbf{R}_\infty^n).$$

Conversely, suppose that (4.3) is true. Then

$$E_2 f, E_2 D_{X_k} f, E_2 D_{X_i} D_{X_k} f, E_4 D_t f \in \mathcal{B}_p^{\alpha-2, \frac{1}{2}\alpha-1}(\mathbf{R}^{n+1}).$$

By (4.2) and Corollary 3.5, we have $E_2 f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^{n+1})$. Hence $E_2 f|_{\mathbf{R}_\infty^n} = f \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\Omega)$. \square

Remark 4.4. Let $\alpha \geq 1$. If $u \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$, $0 < T \leq \infty$. Combining Theorem 4.2 and Theorem 4.3 we obtain the estimate

$$(4.4) \quad \|D_X u\|_{\mathcal{B}_p^{\alpha-1, \frac{1}{2}\alpha-\frac{1}{2}}(\mathbf{R}_T^n)} \lesssim \|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)}.$$

5. PROOFS OF THEOREM 1.1

In this section, we study the relation of usual Besov spaces $\mathcal{B}_p^{\alpha-\frac{2}{p}}(\mathbf{R}^n)$ and parabolic Besov spaces $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}_T^n)$.

Theorem 5.1. Let $0 < T < \infty$. Let $f \in \mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)$ and u be defined by (1.1). Then, for $1 \leq p \leq \infty$, we have

$$(5.1) \quad \|u\|_{L^p(\mathbf{R}_T^n)} \lesssim \|f\|_{\mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)}.$$

(Compare with the section 1.8.1 in [22]).

We introduce a function $\phi' \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz space in \mathbf{R}^n , such that

$$\begin{cases} \hat{\phi}'(\xi) > 0, & \text{on } 2^{-1} < |\xi| < 2, \\ \hat{\phi}'(\xi) = 0, & \text{elsewhere,} \end{cases}$$

$$\sum_{-\infty < i < \infty} \hat{\phi}'(2^{-i}\xi) = 1, \quad (\xi \neq 0).$$

We define functions $\phi'_i, \psi' \in \mathcal{S}(\mathbf{R}^n)$ whose Fourier transforms are written by

$$(5.2) \quad \begin{aligned} \hat{\phi}'_i(\xi) &= \hat{\phi}'(2^{-i}\xi), \quad i = 0, \pm 1, \pm 2, \dots, \\ \hat{\psi}'(\xi) &= 1 - \sum_{1 \leq i < \infty} \hat{\phi}'(2^{-i}\xi). \end{aligned}$$

As we defined the parabolic Besov space, we define a Besov space in \mathbf{R}^n . For $\alpha \in \mathbf{R}$ we define the Besov space $\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^n)$ by

$$\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) \mid \|f\|_{\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}} < \infty\}$$

with the norms

$$\begin{aligned} \|f\|_{\mathcal{B}_{pq}^{\alpha, \frac{1}{2}\alpha}} &:= \|\psi' * f\|_{L^p} + \left(\sum_{1 \leq i < \infty} (2^{\alpha i} \|\phi'_i * f\|_{L^p})^q \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, \\ \|f\|_{\mathcal{B}_{p\infty}^{\alpha, \frac{1}{2}\alpha}} &:= \sup(\|\psi' * f\|_{L^p}, 2^{\alpha i} \|\phi'_i * f\|_{L^p}), \end{aligned}$$

where $*$ is a convolution in \mathbf{R}^n . When $p = q$, we simply denote $\mathcal{B}_{pp}^{\alpha, \frac{1}{2}\alpha}$ by $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}$.

Lemma 5.2. *Let $\hat{\Psi}'(\xi) = \hat{\psi}'(\xi) + \hat{\phi}'(2^{-1}\xi) + \hat{\phi}'(2^{-2}\xi)$ and $\hat{\Phi}'(\xi) = \hat{\phi}'(2^{-1}\xi) + \hat{\phi}'(\xi) + \hat{\phi}'(2\xi)$. Let $\hat{\Phi}'_i(\xi) = \hat{\Phi}'(2^{-i}\xi)$, $i \geq 2$ and let $\rho_{ti}(\xi) = \hat{\Phi}'_i(\xi)e^{-t|\xi|^2}$ for each integer $i \geq 2$. Then, $\rho_{ti}(\xi)s'$ are $L^p(\mathbf{R}^n)$ -multipliers with norms $M(t, i)$ for $1 \leq p \leq \infty$. Furthermore, for $t > 0$*

$$(5.3) \quad M(t, i) \lesssim e^{-\frac{1}{4}t2^{2i}} \sum_{0 \leq l \leq L} t^l 2^{2il} \lesssim e^{-\frac{1}{8}t2^{2i}},$$

where $L = [\frac{n}{2}] + 1$.

Proof. Let $t > 0$. The $L^p(\mathbf{R}^n)$ -multiplier norms $M(t, i)$ of $\rho_{ti}(\xi)$ are equal to $L^p(\mathbf{R}^n)$ -multiplier norms of $\rho'_{ti}(\xi) = \hat{\Phi}'(\xi)e^{-t2^{2i}|\xi|^2}$ (see Theorem 6.1.3 in [2]). To prove our lemma, we make use of the Lemma 6.1.5 in [2]. Let $\beta = (\beta_1, \dots, \beta_n)$, where β_i are non-negative integers. Then, we have

$$|D_\xi^\beta \rho'_{ti}(\xi)| \lesssim e^{-\frac{1}{4}t2^{2i}} \sum_{0 \leq l \leq |\beta|} t^l 2^{2il} \chi_{\frac{1}{4} < |\xi| < 4}(\xi),$$

where χ is a characteristic function. Let $L = [\frac{n}{2}] + 1$ and $\theta = \frac{n}{2L}$. Then by Lemma 6.1.5 in [2], the $L^p(\mathbf{R}^n)$ -multiplier norms of ρ'_{ti} are dominated by

$$\|\rho'_{ti}\|_{L^2(\mathbf{R}^n)}^{1-\theta} \sup_{|\beta|=L} \|D^\beta \rho'_{ti}\|_{L^2(\mathbf{R}^n)}^\theta \lesssim e^{-\frac{1}{4}t2^{2i}} \sum_{0 \leq l \leq L} t^l 2^{2il}.$$

This completes the proof. \square

Proof of Theorem 5.1. Since the proof is similar, we only show in the case $1 \leq p < \infty$. To prove Theorem 5.1, we use $\hat{\psi}'(\xi) + \sum_{1 \leq i < \infty} \hat{\phi}'(2^{-i}\xi) = 1$ for all $\xi \in \mathbf{R}^n$. Note that

$$\hat{u}(\xi, t) = (\hat{\Psi}'(\xi)\hat{\psi}'(\xi) + \hat{\Psi}'(\xi)\hat{\phi}'(2^{-1}\xi))e^{-t|\xi|^2}\hat{f} + \sum_{i=2}^{\infty} \hat{\Phi}'(2^{-i}\xi)\hat{\phi}'(2^{-i}\xi)e^{-t|\xi|^2}\hat{f}.$$

where \hat{u} is the Fourier transform in \mathbf{R}^n . Hence, we have

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^n} |u(X, t)|^p dX dt &\leq c_p \int_0^T \int_{\mathbf{R}^n} |\mathcal{F}^{-1}((\hat{\Psi}'(\xi)\hat{\psi}'(\xi) + \hat{\Psi}'(\xi)\hat{\phi}'_1(\xi))e^{-t|\xi|^2}\hat{f})|^p dX dt \\ &\quad + c_p \int_0^T \int_{\mathbf{R}^n} |\mathcal{F}^{-1}(\sum_{2 \leq i < \infty} \hat{\Phi}'_i(\xi)e^{-t|\xi|^2}\hat{\phi}'_i(\xi)\hat{f})|^p dX dt. \end{aligned}$$

Note that by Young's inequality, we have

$$(5.4) \quad \int_{\mathbf{R}^n} |\Gamma(\cdot, t) * \Psi'| dX \leq \int_{\mathbf{R}^n} |\Psi'(X)| dX < \infty,$$

Applying Young's inequality again, the first term is dominated by

$$(5.5) \quad \int_0^T (\|f * \psi'\|_{L^p(\mathbf{R}^n)}^p + \|f * \phi'_1\|_{L^p(\mathbf{R}^n)}^p) dt.$$

Since $\Phi'_i(\xi)e^{-t|\xi|^2}$ are $L^p(\mathbf{R}^n)$ -multipliers with norms $M(t, i)$ (see lemma 5.2), we have

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}^n} |\mathcal{F}^{-1} \left(\sum_{1 \leq i < \infty} \psi''_i(\xi) e^{-t|\xi|^2} \phi'_i(\xi) \hat{f} \right)|^p dX dt \\ & \leq \int_0^T \left(\sum_{t2^{2i} \leq 1} M(t, i) \|f * \phi'_i\|_{L^p} \right)^p dt \\ & \quad + \int_0^T \left(\sum_{t2^{2i} \geq 1} M(t, i) \|f * \phi'_i\|_{L^p} \right)^p dt \\ & = I_1 + I_2. \end{aligned}$$

By Lemma 5.2, for $t2^{2i} \leq 1$, we have $M(t, i) \lesssim t^{\alpha - \frac{2}{p}}$. Since $\alpha < \frac{2}{p}$, we take $a \in \mathbf{R}$ satisfying $\alpha - \frac{2}{p} < a < 0$ and using Hölder inequality, we have

$$\begin{aligned} I_1 & \lesssim \int_0^T \left(\sum_{t2^{2i} \leq 1} 2^{-\frac{p}{p-1}ai} \right)^{p-1} \sum_{t2^{2i} \leq 1} 2^{pai} \|f * \phi'_i\|_{L^p}^p dt \\ & \lesssim \int_0^T t^{\frac{1}{2}pa} \sum_{t2^{2i} \leq 1} 2^{pai} \|f * \phi'_i\|_{L^p}^p dt \\ & \lesssim \sum_{1 \leq i < \infty} 2^{pai} \|f * \phi'_i\|_{L^p}^p \int_0^{2^{-2i}} t^{\frac{1}{2}pa} dt \\ & = c \sum_{1 \leq i < \infty} 2^{-2i} \|f * \phi'_i\|_{L^p}^p. \end{aligned}$$

Now, we estimate I_2 . By Lemma 5.2, we have that $M(t, i) \lesssim (t2^{2i})^{-m} \sum_{0 \leq i \leq L} t^i 2^{2ii} \lesssim 2^{(2L-2m)i} t^{L-m}$ for $t2^{2i} \geq 1$ and $m > 0$. Let us take m and b satisfying $b > 0$ and $\frac{p}{2}(2L -$

$2m) + \frac{1}{2}pb + 1 < 0$. Then, we get

$$\begin{aligned}
I_2 &\lesssim \int_0^T \left(\sum_{t2^{2i} \geq 1} 2^{(2L-2m)i} t^{L-m} \|f * \phi'_i\|_{L^p} \right)^p dt \\
&\lesssim \int_0^\infty t^{\frac{p}{2}(2L-2m)} \left(\sum_{t2^{2i} \geq 1} 2^{-\frac{p}{p-1}bi} \right)^{p-1} \sum_{t2^{2i} \geq 1} 2^{pbi} 2^{p(2L-2m)i} \|f * \phi'_i\|_{L^p}^p dt \\
&\lesssim \int_0^\infty t^{\frac{p}{2}(2L-2m) + \frac{1}{2}pb} \sum_{t2^{2i} \geq 1} 2^{pbi} 2^{p(2L-2m)i} \|f * \phi'_i\|_{L^p}^p dt \\
&\lesssim \sum_{1 \leq i < \infty} 2^{pbi} 2^{p(2L-2m)i} \|f * \phi'_i\|_{L^p}^p \int_{2^{-2i}}^\infty t^{\frac{p}{2}(2L-2m) + \frac{1}{2}pb} dt \\
&= c \sum_{1 \leq i < \infty} 2^{-2i} \|f * \phi'_i\|_{L^p}^p.
\end{aligned}$$

Hence, we complete the proof of theorem 5.1. \square

Theorem 5.3. *Let $1 \leq p \leq \infty$ and i be a non-negative integer. Let $f \in \mathcal{B}^{2i-\frac{2}{p}}(\mathbf{R}^n)$ and u be defined by (1.1). Then, for $T > 0$, we have*

$$(5.6) \quad \|u\|_{W_p^{2i,i}(\mathbf{R}_T^n)} \lesssim \|f\|_{\mathcal{B}_p^{2i-\frac{2}{p}}(\mathbf{R}^n)}.$$

Proof. From Theorem 5.1, (5.6) holds for $i = 0$.

Let $i > 0$. We denote $\Delta = \sum_{1 \leq k \leq n} D_{X_k}^2$ and $\Delta^{l+1} = \Delta \Delta^l$ for $l \geq 2$. Since $D_t^l D_X^\beta u(X, t) = \Delta^l D_X^\beta u(X, t) = \langle \Delta^l D_X^\beta f, \Gamma(X - \cdot, t) \rangle$ for $|\beta| + 2l \leq 2i$, by Theorem 5.1, we have

$$\|D_t^l D_X^\beta u\|_{L^p(\mathbf{R}_T^n)} \lesssim \|\Delta^l D_X^\beta f\|_{\mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)} \lesssim \|f\|_{\mathcal{B}_p^{2i-\frac{2}{p}}(\mathbf{R}^n)}.$$

For the last inequality, we used the well-known fact

$$(5.7) \quad \|f\|_{\mathcal{B}_p^\alpha(\mathbf{R}^n)} \approx \|f\|_{\mathcal{B}_p^{\alpha-1}(\mathbf{R}^n)} + \|D_X f\|_{\mathcal{B}_p^{\alpha-1}(\mathbf{R}^n)}$$

for each $\alpha \in \mathbf{R}$ and $1 \leq p \leq \infty$ (see [2]). This completes the proof of Theorem 5.3. \square

In fact, for $i \geq 1$, $1 < p < \infty$ the Theorem 5.3 is known result before (see [13]).

Theorem 5.4. *Let $f \in \mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)$ and u be defined by (1.1). Then, for $1 \leq p \leq \infty$,*

$$(5.8) \quad \|f\|_{\mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)} \lesssim \|u\|_{L^p(\mathbf{R}_T^n)}.$$

Proof. Since the proof of the case $p = \infty$ is similar, we only prove the case $1 \leq p < \infty$. Note that the $L^p(\mathbf{R}^n)$ -multiplier norms of $\hat{\phi}'(2^{-i}\xi)e^{|2^{-i}\xi|^2}$ are equal to the $L^p(\mathbf{R}^n)$ -multiplier norm of $\hat{\phi}'(\xi)e^{|\xi|^2}$, where $\hat{\phi}'$ is defined in (5.2) (see Theorem 6.1.3 in [2]). Using Lemma 6.1.5

in [2], we have the $L^p(\mathbf{R}^n)$ -multiplier norm of $\hat{\phi}'(\xi)e^{|\xi|^2}$ is finite. Hence, for $1 \leq p < \infty$, we have

$$\begin{aligned}
(2^{-\frac{2}{p}i} \|f * \phi'_i\|_{L^p(\mathbf{R}^n)})^p &= 2^{-2i} \int_{\mathbf{R}^n} |\mathcal{F}^{-1}(\hat{\phi}'(2^{-i}\xi)e^{2^{-2i}|\xi|^2}e^{-2^{-2i}|\xi|^2}\hat{f})|^p dX \\
&\lesssim 2^{-2i} \int_{\mathbf{R}^n} |u(X, 2^{-2i})|^p dX \\
&\lesssim \int_{2^{-2i}}^{2^{-2i+2}} \int_{\mathbf{R}^n} |u(X, 2^{-2i})|^p dX dt \\
&\lesssim \int_{2^{-2i}}^{2^{-2i+2}} \int_{\mathbf{R}^n} (2^{-i(n+2)} \int_{J_{2^{-i-1}}(X, 2^{-2i})} u(Y, s) dY ds)^p dX dt \\
&\lesssim \int_{2^{-2i}}^{2^{-2i+2}} \int_{\mathbf{R}^n} |u(X, t)|^p dX dt,
\end{aligned}$$

where $J_r(X, t) = \{(Y, s) \in \mathbf{R}^{n+1} \mid |X - Y| < r, |t - s|^{\frac{1}{2}} < r\}$. Hence, we have

$$\begin{aligned}
\sum_{1 \leq i < \infty} (2^{-\frac{2}{p}i} \|f * \phi'_i\|_{L^p(\mathbf{R}^n)})^p &\lesssim \sum_{1 \leq i < \infty} \int_{2^{-2i}}^{2^{-2i+2}} \int_{\mathbf{R}^n} |u(X, t)|^p dX dt \\
&\lesssim \int_0^1 \int_{\mathbf{R}^n} |u(X, t)|^p dX dt \\
&\lesssim \int_0^1 \int_{\mathbf{R}^n} |u(X, t)|^p dX dt.
\end{aligned}$$

Similarly, the $L^p(\mathbf{R}^n)$ -multiplier of $\hat{\psi}'(\xi)e^{\frac{1}{2}|\xi|^2}$ is finite. Hence, we have

$$\begin{aligned}
\|f * \psi'\|_{L^p(\mathbf{R}^n)}^p &\lesssim \int_{\mathbf{R}^n} |\mathcal{F}^{-1}(\psi'e^{\frac{1}{2}|\xi|^2}e^{-\frac{1}{2}|\xi|^2}\hat{f})|^p \\
&\lesssim \int_{\mathbf{R}^n} |u(X, \frac{1}{2})|^p dX \\
&\lesssim \int_{\mathbf{R}^n} \int_{J_{\frac{1}{4}}(X, \frac{1}{2})} |u(Y, s)|^p dY ds dX \\
&\lesssim \int_0^1 \int_{\mathbf{R}^n} |u(Y, s)|^p dY ds.
\end{aligned}$$

Hence, we proved Theorem 5.4 when $T = 1$. For general $T > 0$, we use scaling. Note that

$$v(X, t) = u(T^{\frac{1}{2}}X, Tt) = \int_{\mathbf{R}^n} \Gamma(X - Y, t) f_T(Y) dY,$$

where $f_T(Y) = f(T^{\frac{1}{2}}Y)$. Hence, we have

$$\|f_T\|_{\mathcal{B}_p^{-\frac{2}{p}}} \lesssim \int_0^1 \int_{\mathbf{R}^n} |v(X, t)|^p dX dt = cT^{-\frac{n+2}{2}} \int_0^T \int_{\mathbf{R}^n} |u(X, t)|^p dX dt.$$

Since $\|f\|_{\mathcal{B}_p^{-\frac{2}{p}}} \lesssim_T \|f_T\|_{\mathcal{B}_p^{-\frac{2}{p}}}$, we obtain Theorem 5.4 for general $0 < T < \infty$.

□

Theorem 5.5. *Let $1 \leq p \leq \infty$ and i be a non-negative integer. Let $f \in \mathcal{B}_p^{2i-\frac{2}{p}}(\mathbf{R}^n)$ and u is defined by (1.1). Then*

$$(5.9) \quad \|f\|_{\mathcal{B}_p^{2i-\frac{2}{p}}(\mathbf{R}^n)} \lesssim \|u\|_{W_p^{2i,i}(\mathbf{R}_T^n)}.$$

Proof. In Theorem 5.4, we have (5.9) for $i = 0$. Let $i > 0$. Notice that for $|\beta| \leq 2i$, we have $D_X^\beta u(X, t) = c_n \int_{\mathbf{R}^n} t^{-\frac{n}{2}} e^{-\frac{|X-Y|^2}{4t}} D_Y^\beta f(Y) dY$. By (5.7) and (5.8), we have

$$\|f\|_{\mathcal{B}_p^{2i-\frac{2}{p}}(\mathbf{R}^n)} \lesssim \sum_{|\beta| \leq 2i} \|D_X^\beta f\|_{\mathcal{B}_p^{-\frac{2}{p}}(\mathbf{R}^n)} \lesssim \sum_{|\beta|+2l \leq 2i} \|D_X^\beta u\|_{L^p(\mathbf{R}_T^{n+1})} \lesssim \|u\|_{W_p^{2i,i}(\mathbf{R}_T^n)}.$$

This completes the proof of Theorem 5.5. \square

Combining Theorem 5.1-Theorem 5.5 and by the real interpolation property, we obtain the result of Theorem 1.1.

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